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Reliability-Based Topology Optimization with Uncertainties

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This research proposes a reliability-based topology optimization (RBTO) using the finite element method. RBTO is a topology optimization based on probabilistic (or reliability) constraints. Young's modulus, thickness, and loading are considered as the uncertain variables and RBTO is applied to static and eigenvalue problems. The RBTO problems are formulated and a sensitivity analysis is performed. In order to compute probability constraints, two methods — RIA and PMA — are used. Several examples show the effectiveness of the proposed method by comparing the classical safety factor method.

Key Words : Reliability-based Design Optimization (RBDO), Topology Optimization, Reliability-based Topology Optimization (RBTO), Uncertainty

Nomenclature

- $a(\cdot,\cdot)$: Strain energy bilinear form
- $d(\cdot,\cdot)$: Kinetic energy bilinear form
- $l(\cdot)$: Load linear form
- $\mathbf{C}$: Elasticity tensor
- $\mathbf{C}_{ad}$: Admissible rigidity tensors
- E : Young's modulus of the given isotropic material
- G : Limit state function (performance function)
- F : Force
- P_s : System probability for the success
- P_t : Target probability for the success
- t : Thickness
- X : Random variable (Uncertain variable)
- $\mathbf{y}$: Eigenvector
- Z : Space of kinematically admissible displacement fields
- β_s : System reliability index for the success
- β_t : Target reliability index for the success
- ζ : i^{th} Eigenvalue of the system
- f_{ni} : i^{th} natural frequency of the system
- v : Poisson's ratio of the given isotropic material
- η_i : i^{th} density function (design variable)

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1. Introduction

The goal of probabilistic optimization is to consider the variations of performances which are caused by uncertainties, as these uncertain variables have variances on certain design points. In deterministic optimization, these uncertainties are not considered. Thus, deterministic optimum designs can be unreliable with regard to design failures.

In probabilistic optimization, cost minimization and bringing probabilistic constraints on target should be done simultaneously. Considerable research on probabilistic optimization has been done in order to improve the quality of a product by minimizing the effects of variation. The main difference between the deterministic optimization and RBDO are their constraints, as reliability-based design optimization (RBDO) (Thanedar and Kodiyalam, 1992 ; Chandu and Grandhi, 1995 ; Haldar and Mahadevan, 2000) has the same objective as deterministic optimization. However, in RBDO, probabilistic constraints are formulated so as to construct approximated linear (or quadratic) functions to closely represent the nonlinear limit state functions for the reliability index (or safety index) calculation and optimization by using the appropriate transformations.

Topology optimization (Bendsoe and Sigmund, 2003) is mostly used for the initial design of products, while other conventional methods such as sizing or shape/configuration optimization focus on improving the current design. Since topology optimization was introduced a decade ago, copious research has been done in the fields of engineering and mathematics. Recently, topology optimization has been applied to multi-physics fields such as MEMS (micro-electro-mechanical systems) and practical industrial problems.

In this research, a reliability-based topology optimization (RBTO) using the finite element method is proposed. RBTO is a topology optimization based on probabilistic (or reliability) constraints. Young's modulus, thickness, and loading are considered as the uncertain variables. A majority of researchers use the reliability index as a probabilistic constraint in RBDO. This approach is called the reliability index approach (RIA). Recently, a performance measure approach (PMA) (Tu et al., 1999 ; 2001) has been proposed for RBDO in order to evaluate the probabilistic constraint in an inverse reliability analysis which is consistent with the conventional RIA. The two methods, RIA and PMA, are used in order to compute the probability constraints, as they are both representative analysis methods of probabilistic constraints in RBDO.

RBTO is then applied to static and eigenvalue problems to show the effectiveness of the proposed method. The limit state function is linearly approximated at each iteration in order to evaluate the probabilistic constraints. This approximation method is called the first-order reliability method (FORM), which is widely used in RBDO research.

This paper consists of six parts. First, the general formulation of topology optimization is presented. Then, the efficient methodology for evaluating probabilistic constraints is given and RBTO is formulated in the following chapter. Next, the sensitivity analysis of RBTO is explained. For the implementation, a commercial optimizer, DOT, is used as an optimizer and a commercial FEA code, ANSYS, is used as an analyzer. For reliability and sensitivity evaluation, subroutines are developed by using the Visual C++ language. ANSTOP, an in-house code for topology optimization based on ANSYS, is used for pre- and post-processing. Some numerical examples are then presented. The summary, conclusion, and recommendations are given in the final chapter.

2. Formulation of Topology Optimization

The purpose of topology optimization is to find the optimal layout of a structure within a specified region. The only known quantities in the problem are the applied load, the possible support conditions, the volume of the structure to be used as well as some additional design restrictions, such as the location and size of the prescribed holes. In this problem, the physical size, shape and connectivity of the structure are unknown. Thus, the topology, shape, and size of the structure are represented by a set of distributed functions defined on a fixed design domain, rather than by the standard parametric functions.

2.1 Formulation of topology optimization for static problems

The general set-up for static problems is described by the following situation. Consider a mechanical element as a body occupying a domain, Ω^m, which is a part of a larger reference domain, Ω, in R^2. The reference domain Ω is chosen so as to allow for the definition of applied loads and boundary conditions. By referring to the reference domain Ω, we can define the optimal choice of elasticity tensor, $C(x)^*$, that is variable over the domain. By introducing the energy bilinear form, $a(\mathbf{z}, \bar{\mathbf{z}})$, which is the internal virtual work of an elastic body at equilibrium, $\mathbf{z}$, and arbitrary virtual displacement, $\bar{\mathbf{z}}$, and the load linear form, $l(\bar{\mathbf{z}})$, the minimum displacement (maximum local stiffness) problem takes the form :

$$\begin{aligned} &\textit{Minimize} \quad \psi(\mathbf{z}) \\ &\textit{Subject to} \quad a(\mathbf{z}, \bar{\mathbf{z}}) = l(\bar{\mathbf{z}}), \text{ for all } \bar{\mathbf{z}} \in Z \\ &\qquad\qquad\quad \mathbf{C} \in \mathbf{C}_{ad} \end{aligned} \tag{1}$$

where, $\psi(\mathbf{z})$ is the objective function, such as displacement, compliance. Z denotes the space of kinematically admissible displacement fields.

$$a(\mathbf{z}, \bar{\mathbf{z}}) = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{z})^T \mathbf{C}(x) \boldsymbol{\varepsilon}(\bar{\mathbf{z}}) \, d\Omega \tag{2}$$

$$l(\bar{\mathbf{z}}) = \int_{\Omega} \mathbf{f}^T \bar{\mathbf{z}} d\Omega + \int_{\Gamma_T} \mathbf{t}^T \bar{\mathbf{z}} ds \tag{3}$$

where, $\mathbf{C}(x)$ is the elasticity tensor, and the equilibrium equation is written in its weak, variational form. $\mathbf{f}$ is the body force, and $\mathbf{t}$ is the boundary traction on the traction boundary, Γ_T.

In Eq. (2), the component of the linearized strain, $\varepsilon_{ij}(\mathbf{z})$, is :

$$\varepsilon_{ij}(\mathbf{z}) = \frac{1}{2}\left(\frac{\partial z_i}{\partial x_j} + \frac{\partial z_j}{\partial x_i}\right) \tag{4}$$

where, z_i is the displacement component of the i^{th} coordinate direction, and x_i is the i^{th} coordinate.

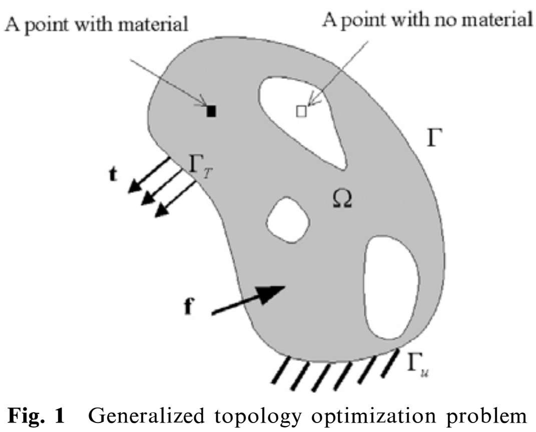


Fig. 1 Generalized topology optimization problem

In Eq. (1), $\mathbf{C}_{ad}$ refers to the set of admissible rigidity tensors for the design problem. With regards to topology design, $\mathbf{C}_{ad}$ consists of all rigidity tensors of a given isotropic material and zero properties elsewhere, the limit of resources being expressed as $\int_{\Omega^m} 1 d\Omega \leq V$. Figure 1 shows the general topology optimization problem.

2.2 Formulation of topology optimization for eigenvalue problems

Eigenvalue optimization is a problem of particular interest in structural design and the topology optimization for this type of problem has been widely used. Eigenvalue structural optimization problems are typically formulated where the design variables include the constitutive tensors that characterize material properties, as static problems. In general Eigenvalue optimization, the objective is either maximizing or minimizing a certain eigenvalue of the structure in free vibration.

Researchers commonly consider a design problem in two- or three-dimensional elasticity where the elastic properties of the medium are represented by the constitutive tensor $\mathbf{C}$. This design process maximizes or minimizes a certain eigenvalue of the structure.

The general formulation for an eigenvalue problem is as follows :

$$\begin{aligned} &\textit{Minimize} \quad \zeta_i \\ &\textit{Subject to} \quad a(\mathbf{y}, \bar{\mathbf{y}}) = \zeta d(\mathbf{y}, \bar{\mathbf{y}}), \\ &\qquad\qquad\quad \text{for all } \bar{\mathbf{y}} \in Z \\ &\qquad\qquad\quad \mathbf{C} \in \mathbf{C}_{ad} \end{aligned} \tag{5}$$

where, $\zeta_i = \omega_i^2 = 4\pi^2 f_{ni}^2$ is the specified eigenvalue of the system, and $\mathbf{y}$ is the eigenvector for the eigenvalue, ζ_i.

The kinetic energy bilinear form, $d(\mathbf{y}, \bar{\mathbf{y}})$, is written as :

$$d(\mathbf{y}, \bar{\mathbf{y}}) = \int_{\Omega} \rho \mathbf{y}^T \bar{\mathbf{y}} d\Omega \tag{6}$$

where, ρ is the density of the structure.

2.3 Density method

Homogenization and density methods are popular methods of solving topology optimization problems. In this research, the density method is used. The design problem in the density method can be formulated as a sizing problem by modifying the stiffness matrix so that it continuously depends on an artificial density-like function. This function is then regarded as the design variable. However, the primary requirement for this formulation is that the optimization results in designs that consist almost entirely of regions of material or no material. This means that the intermediate values of this artificial density function should be penalized in a manner analogous to other continuous optimization approximations of 0-1 problems. Therefore, the density method typically defines the relationships between the design variables and the materials as :

$$\begin{aligned} &\eta(x) \in L^{\infty}(\Omega) \\ &\mathbf{C}(x) = \eta^{p}\mathbf{C}^{0},\ p>1 \\ &\int_{\Omega} \eta(x)\, d\Omega \le V\ ;\ 0 \le \eta(x) \le 1,\ x \in \Omega \end{aligned} \tag{7}$$

where, p is the penalization factor, η is the density function, and $L^{\infty}(\Omega)$ is the essentially bounded function (Lebesgue-measurable function). The penalization factor used in this paper is 3 like the conventional structural cases. $\mathbf{C}^0$ stands for the isotropic material property of the given design, which is defined as :

$$\mathbf{C}^{0} = \frac{E}{1-v^{2}} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{8}$$

E and v are Young's modulus and Poisson's ratio of the given isotropic material.

Moreover, the topology designs not only exhibit dependence on the value of the penalization factor, p, but also on the finite element mesh applied. Nevertheless, because the density method can construct acceptable results and it is easy to apply, this method is widely used.

3. Formulation of Reliability-Based Topology Optimization

In this research, the minimization of volume is the objective function of Reliability-based Topology Optimization (RBTO). A displacement and a first Eigenvalue are considered as the limit-state function for the static and Eigenvalue problems, respectively. Design variables are the density functions, η_i, in each finite element.

Young's modulus, thickness, and loading are considered as uncertain variables. All uncertain variables are assumed to be normal random variables.

3.1 Formulation of RBTO for static problems

The general form of RBTO for static problems is described as follows :

Find the design variable vector $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ such that

$$\begin{aligned} &\textit{Minimize} && \textit{Total Volume } V(\eta_i) \\ &\textit{Subject to} && P_s(X) = P[G(\eta, X_j)] \ge P_t \\ & && 0 \le \eta_i \le 1 \\ & && i = 1, \cdots, ndv \text{ and } j = 1, 2, 3 \end{aligned} \tag{9}$$

where

$$G = -\psi + \psi_{\max} \ge 0 \tag{10}$$

$$\psi \equiv z(\hat{x}) = \int_{\Omega} \hat{\delta}(x-\hat{x})\, z(x)\, d\Omega \tag{11}$$

ψ is the displacement at an isolated point, $\hat{x}$, and $\hat{\delta}$ is the Dirac-Delta function. X_j is the j[th] uncertain variable. The limit-state, Eq. (10), implies that if the displacement Ψ is larger than the limit value $\psi_{\max}$, the system fails.

Then, using the Reliability Index Approach (RIA) and Performance Measure Approach (PMA) Eq. (9) can be formulated in two ways :

For RIA,

$$\begin{aligned} &\textit{Minimize} && \textit{Total Volume } V(\eta_i) \\ &\textit{Subject to} && \beta_s(\eta_i, X_j) \ge \beta_t \\ & && \textit{when } G(X) = 0 \\ & && \textit{for each evaluation} \\ & && 0 \le \eta_i \le 1 \\ & && i = 1, \cdots, ndv \text{ and } j = 1, 2, 3 \end{aligned} \tag{12}$$

For PMA,

$$\begin{aligned} &\textit{Minimize} && \textit{Total Volume } V(\eta_i) \\ &\textit{Subject to} && G^{*}(\eta_i, X_j) \ge 0 \\ & && \textit{when } \beta_s = \beta_t \end{aligned}$$

$$\begin{aligned}&\textit{for each evaluation}\\&0\le\eta_i\le 1\\&i=1,\cdots,ndv \text{ and } j=1,2,3\end{aligned} \quad (13)$$

where, β_s is system reliability index for the success and β_t is target reliability index for the success.

3.2 Formulation of RBTO for eigenvalue problems

The general form of RBTO for eigenvalue problems is the same as Eq. (9). Also, Eqs. (12) and (13) can be used for RIA and PMA in eigenvalue problems. The limit-state function, G, is defined as:

$$G=f_{n1}-f_{\min}\ge 0 \quad (14)$$

where, f_{n1} is the first natural frequency of the system. As in the static problems, Eq. (14) implies that if the first natural frequency is smaller than the limit value, $f_{\min}$, the system fails.

4. Design Sensitivity Analysis (DSA) of RBTO

2.1 Design sensitivity analysis of probabilistic constraints

Continuum sensitivity analysis is used in this research to calculate the sensitivity of the performance. Since a topology optimization problem generally deals with thousands of design variables, the adjoint variable method (AVM) (Haug et al., 1986) is useful. AVM requires calculations depending on the number of performances. Conversely, a direct method, such as the finite difference method, needs an equal number of calculations to the number of design variables.

For RIA, a sensitivity analysis of the reliability index is needed. An approximated method can be obtained from the definition of the reliability index:

$$\beta=\frac{\mu_Z}{\sigma_Z}=\frac{G(X)}{\sqrt{\sum_{j=1}^{n}\left(\frac{\partial G}{\partial X}\right)^2\sigma_{X_j}^2}} \quad (15)$$

Therefore, the design sensitivity of the reliability index is:

$$\begin{aligned}\frac{\partial\beta}{\partial\eta_i}&=\frac{\partial}{\partial\eta_i}\frac{G(X)}{\sqrt{\sum_{j=1}^{n}\left(\frac{\partial G}{\partial X_j}\right)^2\sigma_{X_j}^2}}\\&=\frac{\left(\frac{\partial G(X)}{\partial\eta_i}\right)\sqrt{\sum_{j=1}^{3}\left(\frac{\partial G}{\partial X_j}\right)^2\sigma_{X_j}^2}-G(X)\left(\frac{\partial}{\partial\eta_i}\sqrt{\sum_{j=1}^{3}\left(\frac{\partial G}{\partial X_j}\right)^2\sigma_{X_j}^2}\right)}{\sum_{j=1}^{n}\left(\frac{\partial G}{\partial X_j}\right)^2\sigma_{X_j}^2}\\&=\frac{\left(\frac{\partial G(X)}{\partial\eta_i}\right)\sqrt{\sum_{j=1}^{3}\left(\frac{\partial G}{\partial X_j}\right)^2\sigma_{X_j}^2}-G(X)\left(\frac{\sum_{j=1}^{3}2\left(\frac{\partial G}{\partial X_j}\right)\sigma_{X_j}^2\frac{\partial^2 G}{\partial\eta_i\partial X_j}}{\sqrt{\sum_{j=1}^{3}\left(\frac{\partial G}{\partial X_j}\right)^2\sigma_{X_j}^2}}\right)}{\sum_{j=1}^{n}\left(\frac{\partial G}{X}\right)^2\sigma_{X_j}^2}\end{aligned} \quad (16)$$

For PMA, the sensitivity evaluation is much easier than in RIA because the probability constraint of PMA is simply $G(X)$ when $\beta=\beta_t$. Therefore, the design sensitivity of PMA becomes:

$$\frac{\partial G^*}{\partial\eta_i}=\frac{\partial}{\partial\eta_i}G(X)|_{\beta=\beta t} \quad (17)$$

4.2 DSA for static problems

For the static case, the structural equation is written as:

$$a_\Omega(\mathbf{z},\bar{\mathbf{z}})=l_\Omega(\bar{\mathbf{z}}) \quad (18)$$

The displacement at an isolated point, $\hat{x}$, is considered as a performance measure.

$$\psi\equiv z(\hat{x})=\int_\Omega\hat{\delta}(x-\hat{x})\mathbf{z}(x)\,d\Omega \quad (19)$$

The material derivative of ψ is:

$$\psi'=\int_\Omega\hat{\delta}(x-\hat{x})\mathbf{z}'d\Omega \quad (20)$$

The adjoint equation is written as:

$$a_\Omega(\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}})=\int_\Omega\hat{\delta}(x-\hat{x})\bar{\boldsymbol{\lambda}}d\Omega \text{ for all } \bar{\boldsymbol{\lambda}}\in Z \quad (21)$$

The design sensitivity expression is:

$$\begin{aligned}\psi'&=l'_\Omega(\boldsymbol{\lambda})-a'_\Omega(\mathbf{z},\boldsymbol{\lambda})\\&=-a'_\Omega(\mathbf{z},\boldsymbol{\lambda})\end{aligned} \quad (22)$$

where, λ is the solution of the adjoint equation.

Using Eq. (22), the derivatives of G with respect to the three uncertain variables, which are Young's modulus, E, the thickness, t, and a single force, F, are derived as:

$$\frac{\partial G}{\partial X_1}=\frac{\partial G}{\partial E}=-\left(-\int_\Omega \boldsymbol{\varepsilon}^T\frac{\partial \mathbf{E}}{\partial E}\boldsymbol{\varepsilon}_\lambda d\Omega\right)$$
$$=\frac{1}{E}\int_\Omega \boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}_\lambda d\Omega=\frac{1}{E}a_\Omega(\mathbf{z},\lambda) \tag{23}$$

$$\frac{\partial G}{\partial X_2}=\frac{\partial G}{\partial t}=-\left(-\int_\Omega \boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}_\lambda dA\right)$$
$$=\frac{1}{t}\int_\Omega \boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}_\lambda d\Omega=\frac{1}{t}a_\Omega(\mathbf{z},\lambda) \tag{24}$$

$$\frac{\partial G}{\partial X_3}=\frac{\partial G}{\partial F}=-\frac{\partial \Psi}{\partial F}=-\frac{1}{F}a_\Omega(\mathbf{z},\boldsymbol{\lambda}) \tag{25}$$

These equations are valid for the linear system.

Then, the derivatives of $\frac{\partial G}{\partial X_j}$ with respect to the design variables, η_i, are derived as:

$$\frac{\partial^2 G}{\partial\eta_i\partial X_1}=\frac{1}{E}\frac{\partial}{\partial\eta_i}a_o(\mathbf{z},\boldsymbol{\lambda})=-\frac{p}{E\eta_i}a_\Omega(\mathbf{z},\boldsymbol{\lambda}) \tag{26}$$

$$\frac{\partial^2 G}{\partial\eta_i\partial X_2}=\frac{1}{t}\frac{\partial}{\partial\eta_i}a_\Omega(\mathbf{z},\boldsymbol{\lambda})=-\frac{p}{t\eta_i}a_\Omega(\mathbf{z},\boldsymbol{\lambda}) \tag{27}$$

$$\frac{\partial^2 G}{\partial\eta_i\partial X_3}=\frac{\partial}{\partial\eta_i}\left(\frac{\partial G}{\partial F}\right)=\frac{\partial}{\partial\eta_i}\left(-\frac{\delta}{F}\right)=-\frac{1}{F}\frac{\partial\delta}{\partial\eta_i}$$
$$=-\frac{1}{F}(-a'_\Omega(\mathbf{z},\boldsymbol{\lambda}))=\frac{p}{F\eta_i}a_\Omega(\mathbf{z},\boldsymbol{\lambda}) \tag{28}$$

These sensitivity equations are validated by the finite difference method.

4.3 DSA for eigenvalue problems

For the eigenvalue problem, the structural equation is:

$$a_\Omega(\mathbf{y},\bar{\mathbf{y}})=\zeta d_\Omega(\mathbf{y},\bar{\mathbf{y}})\ \text{ for all } \bar{\mathbf{y}}\in Z \tag{29}$$

Since the eigenvector $\mathbf{y}$ is orthonormal relative to the mass matrix, a normalizing condition must be used to uniquely define the eigenvector. In this case, the normalizing condition is $d_\Omega(\mathbf{y},\mathbf{y})=1$.

The bilinear form, $d_\Omega(\mathbf{y},\bar{\mathbf{y}})$, represents the mass effect in vibration problems. In the case of a simple eigenvalue, the eigenvalue ζ is differentiable, as is the corresponding eigenfunction, $\mathbf{y}$.

The first-order sensitivity of Eq. (29) is:

$$[a_\Omega(\mathbf{y},\bar{\mathbf{y}})]'=\zeta' d_\Omega(\mathbf{y},\bar{\mathbf{y}})+\zeta[d_\Omega(\mathbf{y},\bar{\mathbf{y}})]'$$
$$\zeta=4\pi^2 f_n^2 \tag{30}$$

where

$$d\zeta=4\pi^2\cdot 2f_n\cdot df_n$$
$$\frac{df_n}{dX}=\frac{1}{8\pi^2 f_n}\frac{d\zeta}{dX}$$

Since Eq. (30) holds for all $\bar{\mathbf{y}}\in Z$, this equation may be evaluated with $\bar{\mathbf{y}}=\mathbf{y}$. Using the normalizing condition:

$$\zeta'=[a_\Omega(\mathbf{y},\mathbf{y})]'-\zeta[d_\Omega(\mathbf{y},\mathbf{y})]' \tag{31}$$

Using Eq. (31), the derivatives of G with respect to the uncertain variables are derived as:

$$\frac{\partial G}{\partial X_1}=\frac{\partial f_n}{\partial E}=\frac{1}{8\pi^2 f_n}\frac{1}{E}a_\Omega(\mathbf{y},\mathbf{y}) \tag{32}$$

Then, the derivatives of $\frac{\partial G}{\partial X_j}$ with respect to the design variables, η_i, are derived as:

$$\frac{\partial^2 G}{\partial\eta_i\partial X_1}=\frac{1}{8\pi^2 f_n}\frac{1}{E}\frac{\partial}{\partial\eta_i}a_\Omega(\mathbf{y},\mathbf{y})$$
$$=\frac{1}{8\pi^2 f_n}\frac{p}{E\eta_i}a_\Omega(\mathbf{y},\mathbf{y}) \tag{33}$$

4.4 DSA for total volume

The objective function, the total volume, can be written in its discretized form as:

$$V=\sum_i^n \eta_i A_i t_i \tag{34}$$

and the sensitivity is:

$$\frac{\partial V}{\partial\eta_i}=A_i t_i \tag{35}$$

5. Numerical Examples

5.1 Static problem with 1 uncertain variable

The first example is a cantilever plate (16×10) as shown in Fig. 2. The limit-state function is the displacement at Point A which should be smaller than 3.0. The cantilever plate is modeled as two-dimensional shell elements. One end is fixed and the force is applied at Point A. The plate is meshed into 640 elements. The uncertain variable is Young's modulus. The uncertain variable has 10% variance of the mean value and is assumed to be normally distributed. The target reliability index, β_t, is 3.0.

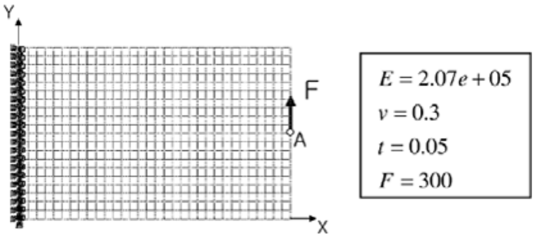
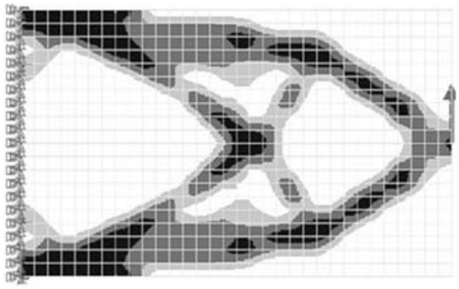


Fig. 2 2-D cantilever plate



(a) RIA

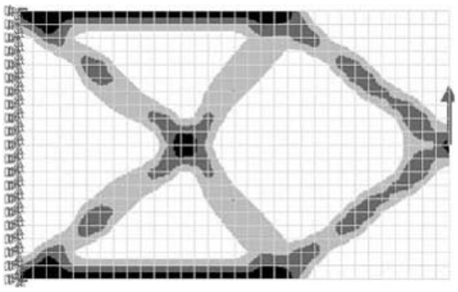
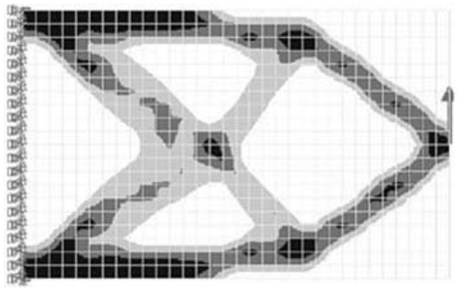


Fig. 3 Deterministic topology optimization (DTO-Static)



(b) PMA

Fig. 4 RBTO with 1 uncertainty (Static)

Table 1 Comparison between DTO and RBTO (Static Case 1)

	Objective (Volume %)	Deflection at the mean value	Reliability
DTO	22.57%	2.99957	0.00159
RBTO with RIA	31.61%	2.10007	3.0005
RBTO with PMA	30.45%	2.10001	2.9998

The deterministic topology optimization (DTO) is written as:

$$\begin{aligned}&\textit{Minimize} \quad \textit{Total Volume}\\&\textit{Subject to} \quad G=-\psi+3.00\geq 0\end{aligned} \tag{36}$$

and the optimal result is shown in Fig. 3.

If RIA is applied, the RBTO problem is written as:

$$\begin{aligned}&\textit{Minimize} \quad \textit{Total Volume}\\&\textit{Subject to} \quad \beta_s\geq 3 \;\textit{when}\; G=-\psi+3.00=0\end{aligned} \tag{37}$$

and the optimal result is shown in Fig. 4(a).

If PMA is applied, the RBTO problem is written as:

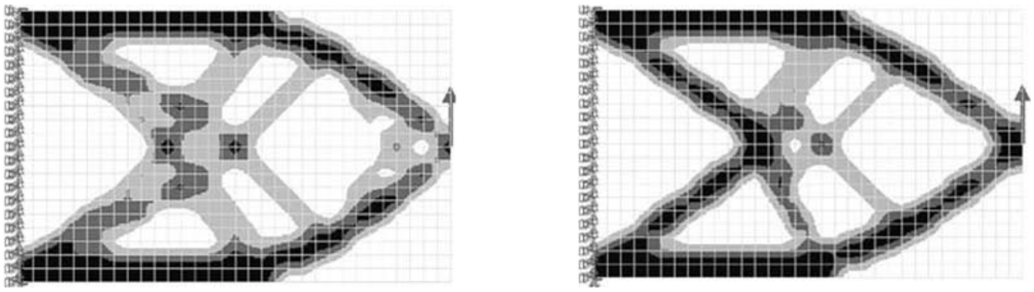
$$\begin{aligned}&\textit{Minimize} \quad \textit{Total Volume}\\&\textit{Subject to} \quad G=-\psi+3.00\geq 0 \;\textit{when}\; \beta_s=3\end{aligned} \tag{38}$$

and the optimal result is shown in Fig. 4(b).

A summary for the static problem with one uncertain variable is shown in Table 1. The objective, the used volume, of the Deterministic Topology Optimization (DTO) is smaller than the results of RBTO. However, DTO has poor reliability, $\beta=0.00159$, which means that the optimum of DTO has about a 50% failure probability. When the reliability is considered into this design, the volume used is more than the DTO required to satisfy the probabilistic constraint. This is because the feasible region becomes smaller due to the distribution of the uncertain variable. RBTO results show that the proposed method achieves the target reliability index. Even though the shapes of Fig. 4(a) and Fig. 4(b) from RIA and PMA are different, the performances (volume and deflection) of the optimal

Table 2 Comparison between DTO and RBTO (Static Case 2)

	Objective (Volume %)	Deflection at the mean value	Reliability
DTO	22.57 %	2.99957	0.00159
RBTO with RIA	35.18%	1.73663	2.99970
RBTO with PMA	34.70%	1.73609	3.00142



(a) RIA (b) PMA

Fig. 5 RBTO with 3 uncertainties (Static)

models from the two methods are quite similar.

5.2 Static problem with 3 uncertain variables

The next example has the same FE model as the first example. However, two more uncertain variables, thickness and loading, are considered as uncertain variables. Again, all uncertain variables have 10% variance of their mean values and are assumed to be normally distributed. The maximum allowable deflection is 3.0 and the target reliability index, β_t, is 3.0. The formulations of RBTO are the same as in the previous example.

The optimum results of RBTO are shown in Fig. 5.

The optimum shapes depicted in Fig. 5 are different from previous results. Because of the increased number of uncertain variables, a more robust solution is obtained in order to satisfy the target reliability. Also, the used volumes are much larger than in the one uncertain variable case.

The trends for the solutions of RIA and PMA are similar for the same conditions. However, there is an important difference. PMA always finds a solution, but RIA sometimes fails to find an optimum due to the divergence problem. Additionally, a complex sensitivity evaluation of the reliability index, β, is required in the case of RIA.

From Table 2, we can find that the results of RBTO have a larger volume than DTO. A DTO model using more volume can be more reliable. To check the efficiency of RBTO, a new DTO problem is formulated as:

$$\begin{aligned} &\textit{Minimize} \quad \psi_A \\ &\textit{Subject to} \quad V \leq 35.63\% \end{aligned} \tag{39}$$

where, V is the used volume in optimal design. The limit value of 35.63% is determined from the results of RBTO and this is a slightly larger volume than the RBTO results. This DTO problem is from the safety factor concept.

The optimal solution of Eq. (39) is shown in Fig. 6.

Table 3 shows the comparison of four optimal designs. DTO result with a large volume has a better reliability than the initial DTO results, still but has a smaller reliability than the target value. Also, the deflection of the DTO model is larger than the RBTO models. Therefore, the RBTO method can give better designs more efficiently than DTO by using the safety factor con-

Table 3 Comparison between DTO and RBTO

	Objective (Volume %)	Deflection at the mean value	Reliability
DTO	22.57%	2.99957	0.00159
RBTO with RIA	35.18%	1.73663	2.99970
RBTO with PMA	34.70%	1.73609	3.00142
DTO (volume used equal to RBTO result)	35.60%	2.09118	2.02320

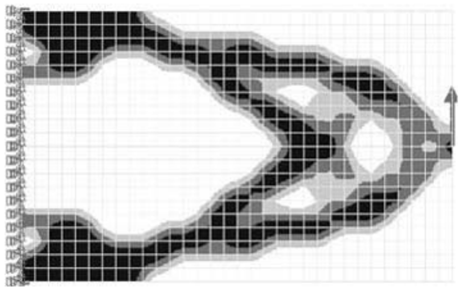


Fig. 6 DTO with the volume equal to RBTO result (Static)

cept. Additionally, an important fact here is that RBTO can effectively obtain a design which satisfies the target probability, whereas the safety factor method is unable to estimate this probability.

5.3 Eigenvalue problem with 1 uncertain variable

The last example is the eigenvalue problem. The objective is to minimize the volume, and the limit-state function is the first natural frequency, which should be larger than 142 Hz. The eigenvalue example model is shown in Fig. 7. A 70× 10 finite element model with shell elements is used, and only x and y direction movements are allowed.

The uncertain variable is Young's modulus. As in prior cases, the uncertain variable has 10% variance of the mean value and it is assumed to be normally distributed.

Initially, the deterministic optimization is written as:

$$\begin{aligned} &\textit{Minimize} \quad \textit{Total Volume} \\ &\textit{Subject to} \quad G=f_{ni}-142.0\geq 0 \end{aligned} \tag{40}$$

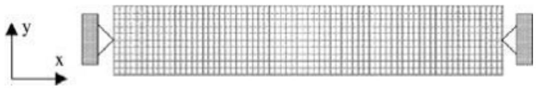


Fig. 7 Eigenvalue Problem Model

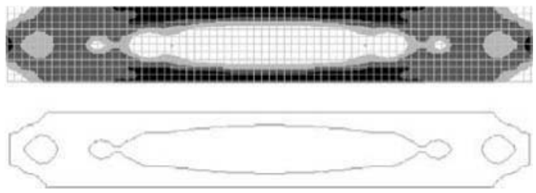


Fig. 8 Contour Plot and Contour Line Plot of DTO (Eigenvalue)

The optimal result is shown in Fig. 8 as contour and contour line plots. A contour line plot is obtained by intersecting the 0.5 density threshold.

If RIA is applied, the RBTO problem is written as:

$$\begin{aligned} &\textit{Minimize} \quad \textit{Total Volume} \\ &\textit{Subject to} \quad \beta_s\geq 3 \textit{ when } G=f_{ni}-142.0=0 \end{aligned} \tag{41}$$

If PMA is applied, the RBTO problem is written as:

$$\begin{aligned} &\textit{Minimize} \quad \textit{Total Volume} \\ &\textit{Subject to} \quad G=f_{n1}-142.0\geq 0 \textit{ when } \beta_s=3 \end{aligned} \tag{42}$$

The RBTO results using RIA and PMA are given in Figs. 9 and 10.

An additional DTO problem that has a 71% volume limit is solved from the safety factor concept. The 71% limit value is from the used volume in RBTO, and the result is shown in Fig. 11.

Table 4 shows results of DTO and RBTO. As in the static case, the last row is the result of DTO with the same volume as RBTO. Also in this case,

Table 4 Comparison between DTO and RBTO (Eigenvalue Case)

	Objective (Volume %)	1st Natural Freq at the mean value	Reliability
DTO	50.574%	142.023	0.003
RBTO with RIA	70.065%	169.712	2.999
RBTO with PMA	71.689%	169.741	3.002
DTO with the same volume as RBTO	70.881%	168.488	2.897

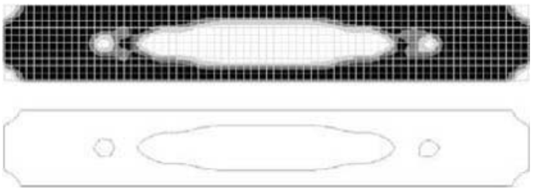


Fig. 9 RBTO with uncertainty (Eigenvalue : RIA)

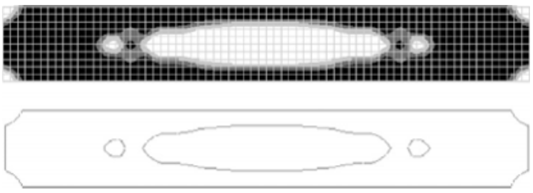


Fig. 10 RBTO with uncertainty (Eigenvalue : PMA)

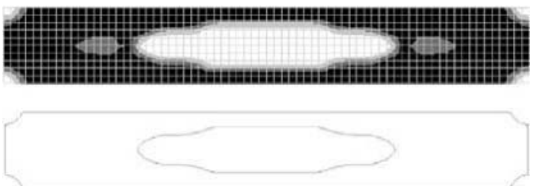


Fig. 11 DTO with the same volume as RBTO (Eigenvalue Case)

DTO could not give better results than RBTO, although the same volume was used. In this example, the reliability of RBTO is slightly better, however, the safety factor method was unable to assess the probability of the performance. RIA and PMA gave almost the same results in this case.

6. Conclusions

In this article, a reliability-based topology optimization (RBTO) is proposed and a RBTO program is implemented by using RIA and PMA approaches. The objective function of RBTO is the minimization of the volume. To estimate the failure probability, a displacement and a first eigenvalue are considered as the limit-state function for static and eigenvalue problems, respectively.

Young's modulus, thickness, and loading are considered as the uncertain variables. All uncertain variables are assumed to be normally distributed random variables. To calculate the probability of the constraints, the first-order reliability method (FORM) is used. In FORM, the limit state function is linearly approximated at each iteration.

2-D problems are solved in order to show the effectiveness of the proposed RBTO. The proposed method gave results that are more reliable with respect to uncertainties. Considering the probability constraints, it is possible to make robust and low-cost products. The original DTO result gives only a 50% success rate probability, but the proposed RBTO can give the requested solution under the condition of several uncertainties. Obviously, the statistically improved design uses more material or volume than the design from DTO. However, even if a DTO problem uses the same volume as RBTO, the DTO solution is not as reliable as RBTO.

Two different approaches, RIA and PMA, were adopted for the RBTO formulation. These two approaches gave slightly different optimal topologies, but their trends were similar for the same parameter conditions. However, in every instance, PMA has the better convergence to find an optimum than RIA. Additionally, RIA requires a complex sensitivity evaluation of the reliability index, β.

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